

Thirring Model as a Gauge Theory*

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Abstract

We reformulate the Thirring model in D ($2 \leq D < 4$) dimensions as a gauge theory by introducing $U(1)$ hidden local symmetry (HLS) and study the dynamical mass generation of the fermion through the Schwinger-Dyson (SD) equation. By virtue of such a gauge symmetry we can greatly simplify the analysis of the SD equation by taking the most appropriate gauge (“nonlocal gauge”) for the HLS. In the case of even-number of (2-component) fermions, we find the dynamical fermion mass generation as the second order phase transition at certain fermion number, which breaks the chiral symmetry but preserves the parity in (2+1) dimensions ($D = 3$). In the infinite four-fermion coupling (massless gauge boson) limit in (2+1) dimensions, the result coincides with that of the (2+1)-dimensional QED, with the critical number of the 4-component fermion being $N_{\text{cr}} = \frac{128}{3\pi^2}$. As to the case of odd-number (2-component) fermion in (2+1) dimensions, the regularization ambiguity on the induced Chern-Simons term may be resolved by specifying the regularization so as to preserve the HLS. Our method also applies to the (1+1) dimensions, the result being consistent with the exact solution. The bosonization mechanism in (1+1) dimensional Thirring model is also reproduced in the context of dual-transformed theory for the HLS.

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§ 1. Introduction

Fermion dynamical mass generation is the central issue of the scenario of the dynamical electroweak symmetry breaking such as the technicolor[1] and the top quark condensate[2]. Special attention has recently been paid to the role of four-fermion interaction in the context of walking technicolor[3], strong ETC technicolor[4] and top quark condensate[2]. These models are based on the Nambu-Jona-Lasinio (NJL) model[5] of the scalar/pseudoscalar-type four-fermion interactions combined with the gauge interactions, so-called gauged NJL model, whose phase structure has been extensively studied through the Schwinger-Dyson (SD) equation (see Ref.[6]). It has been shown[6] that the phase structure of such a gauged NJL model in (3+1) dimensions ($D = 4$) is quite similar to that of the D ($2 < D < 4$) dimensional four-fermion theory of scalar/pseudoscalar-type (without gauge interactions)[7], often called Gross-Neveu model, which is renormalizable in $1/N$ expansion[8].

What about the four-fermion interaction of vector/axialvector-type? Does it also give rise to the fermion mass generation? Of course it can be transformed into the scalar/pseudoscalar-type interaction through the Fierz transformation, but they are independent of each other in the usual framework of $1/N$ expansion (see, e.g., Ref.[9]). This type of four-fermion interaction in fact has been studied in combination with the scalar/pseudoscalar-type (“generalized NJL model”), which is now a popular model as a low energy effective theory of QCD. It is well-known that at the usual $1/N$ leading order in the “generalized NJL model” only the scalar/pseudoscalar four-fermion interaction contributes to the gap equation or to the fermion mass generation, while the vector/axialvector-type four-fermion interaction does not. Thus we may address the following question: If the scalar/pseudoscalar-type four-fermion interaction does not exist at all and the formal $1/N$ leading order is missing in the above gap equation, is the fermion dynamical mass still generated by the vector/axialvector-type alone? If it is the case, such a dynamics would be interesting for the model building beyond the standard model. It would also be interesting if there arises a situation similar to the case of scalar/pseudoscalar-type: Namely, the phase structure of the gauged NJL

model of vector/axialvector-type might have some resemblance to that of the Thirring model[10] (without gauge interactions) in D ($2 < D < 4$) dimensions which is known to be renormalizable in $1/N$ expansion[8, 11].

Thus we wish to study the fermion dynamical mass generation in the D ($2 \leq D < 4$) dimensional Thirring model including the $D = 2$ case. The Thirring model has been extensively studied in (1+1) dimensions since it is explicitly solvable[10]. However, it is only recent that the fermion dynamical mass generation in (2+1)-dimensional Thirring model has been studied by several authors [12, 13, 14, 15, 16]. The results of these papers, however, are different from each other and rather confusing partly due to lack of the discipline of the analysis. In Refs.[12, 13], for example, they introduced vector auxiliary field and pretended it as a gauge field despite the absence of manifest gauge symmetry.

In this paper we reformulate the Thirring model as a gauge theory by introducing the hidden local symmetry (HLS)[9] (see also [17, 11]). When we fix the gauge of HLS to the unitary gauge, we get back to the original Thirring model written in terms of the vector auxiliary field. However, the unitary gauge is notorious for making the actual analysis difficult, while the existence of such a gauge symmetry has some virtues to make the analysis consistent and systematic.

In the case of odd number of 2-component fermions, a peculiarity arises in (2+1) dimensions, namely, the possibility of induced Chern-Simons (CS) term[14]. We may take advantage of existence of the gauge symmetry to resolve the problem of regularization ambiguity concerning the induced CS term; the regularization must be chosen in such a way as to keep the gauge symmetry (Pauli-Villars regularization) as in the (2+1)-dimensional QED (QED₃)[18]. The parity violating CS term will arise from the Pauli-Villars regulator even in the symmetric phase where the fermion mass is not dynamically generated.

As to the case of even number of 2-component fermions, on the other hand, we expect that parity violation in (2+1) dimensions will not be induced, since the above induced CS term of each fermion species can be arranged in pair of opposite sign to cancel each other. We shall demonstrate that an appropriate HLS gauge fixing (nonlocal

gauge[19]) other than the unitary gauge actually leads to the simple and consistent analysis of the SD equation for the D ($2 \leq D < 4$) dimensional Thirring model. The nonlocal gauge is the gauge having no wave function renormalization for the fermion and is the only way to make the bare vertex (ladder) approximation to be consistent with the Ward-Takahashi (WT) identity for the current conservation. We find dynamical chiral symmetry breaking which is parity-conserving in (2+1) dimensions in accord with the Vafa-Witten theorem[20] and establish a second order phase transition at a certain number of 4-component fermions N for each given value of the dimensionless four-fermion coupling g , namely, the critical line on the (N, g) plane. This is somewhat analogous to the existence of the critical N in the QED₃ [21, 22, 23], which has been confirmed by the lattice Monte Carlo simulation [24]. Actually, when the four-fermion coupling constant g goes to infinity (massless gauge boson limit) for $D = 3$, our critical N is explicitly evaluated as $N_{\text{cr}} = \frac{128}{3\pi^2}$ in perfect agreement with that of the QED₃ in the nonlocal gauge[22, 23]. Note that this massless vector limit is smooth thanks to the HLS in contrast to the original Thirring model (unitary gauge) where this limit is singular and ill-defined. For $D = 2$ we explicitly solve the gap equation, which turns out to be consistent with the exact solution.

Although at tree level the HLS gauge boson is merely the auxiliary field, it is a rather common phenomenon that the HLS gauge boson acquires kinetic term by the quantum corrections and hence becomes dynamical [9]. We shall show that the HLS gauge boson also becomes dynamical in the Thirring model once the fermion mass is dynamically generated. In the infinite four-fermion coupling limit this dynamical gauge boson becomes massless and the HLS becomes a spontaneously unbroken gauge symmetry. Were it not for the HLS, on the other hand, this limit becomes ill-defined due to lack of the manifest gauge symmetry [9].

Another advantage of the HLS is that we can use the dual transformation in the Thirring model. Dual transformation is one which manifests the propagating degrees of freedom and maps the theory with strong gauge coupling to that with weak coupling constant [25]. In (1+1) dimensions the HLS together with the dual transformation offers us a straightforward method for the bosonization of the Thirring model in the

context of path integral.

The rest of this paper is organized as follows. In Section 2 we introduce the model in D ($2 \leq D < 4$) dimensions and reformulate it by use of the hidden local symmetry. The nonlocal gauge is introduced at the Lagrangian level rather than in the Schwinger-Dyson equation, which makes the BRS invariance transparent. In Section 3 we study the SD equations under the nonlocal R_ξ gauge fixing for HLS and establish the existence of chiral symmetry breaking dynamical mass generation and of the associated critical line on the (N, g) plane in the case of even number of 2-component fermions. In (2+1) dimensions this mass keeps the parity in accord with the Vafa-Witten theorem. In (1+1) dimensions the fermion mass generation always takes place as far as g is positive. In Section 4 we demonstrate that the gauge boson of the HLS develops a pole due to quantum correction (fermion loop) in the broken phase where the fermion mass is dynamically generated. Section 5 is devoted to the dual transformation of the Thirring model as a new feature of the HLS and to the study of various aspects of the dual transformed theory, particularly the bosonization of Thirring model in (1+1) dimensions. We conclude in Section 6 with some discussions. In Appendix a proof is given of the BRS invariance of the Thirring model Lagrangian with the HLS in the nonlocal gauge.

§ 2. Hidden Local Symmetry

In this section we introduce the HLS [9] into the massless Thirring model in D ($2 \leq D < 4$) dimensions. The Lagrange density of the Thirring model is given by

$$\mathcal{L}_{\text{Thi}} = \sum_a \bar{\Psi}_a i \gamma^\mu \partial_\mu \Psi_a - \frac{G}{2N} \sum_{a,b} (\bar{\Psi}_a \gamma^\mu \Psi_a) (\bar{\Psi}_b \gamma_\mu \Psi_b), \quad (2.1)$$

where Ψ_a is a 4-component Dirac spinor (although formal in D dimensions) and a, b are summed over from 1 to N . Let us rewrite the theory by introducing an auxiliary vector field \tilde{A}_μ :

$$\mathcal{L}' = \sum_a \bar{\Psi}_a i \gamma^\mu \tilde{D}_\mu \Psi_a + \frac{1}{2G} \tilde{A}_\mu \tilde{A}^\mu, \quad (2.2)$$

where $\tilde{D}_\mu = \partial_\mu - \frac{i}{\sqrt{N}}\tilde{A}_\mu$. Note that \tilde{D}_μ is not the covariant derivative in spite of its formal similarity, since the field \tilde{A}^μ is just a vector field which depicts the fermionic current and does not transform as a gauge field. Actually, the Lagrangian (2.2) has no gauge symmetry. It is easy to see that when we solve away the auxiliary field \tilde{A}_μ through the equation of motion for \tilde{A}_μ , Eq.(2.2) is reduced back to the original Thirring model (2.1).

Based on the “ $U(1)/U(1)$ ” nonlinear sigma model, we now show that Eq.(2.2) is gauge equivalent to another model possessing a symmetry $U(1)_{\text{global}} \times U(1)_{\text{local}}$, with the $U(1)_{\text{local}}$ being HLS [9, 17, 11]:

$$\begin{aligned}\mathcal{L}_{\text{HLS}} &= \sum_a \bar{\psi}_a i\gamma^\mu D_\mu \psi_a - \frac{N}{2G} (D_\mu u \cdot u^\dagger)^2 \\ &= \sum_a \bar{\psi}_a i\gamma^\mu D_\mu \psi_a + \frac{1}{2G} (A_\mu - \sqrt{N}\partial_\mu \phi)^2,\end{aligned}\tag{2.3}$$

where

$$u = e^{i\phi}\tag{2.4}$$

and $D_\mu = \partial_\mu - \frac{i}{\sqrt{N}}A_\mu$ is the covariant derivative for A_μ which is a gauge field in contrast to \tilde{A}_μ in Eq.(2.2). It is obvious that Eq.(2.3) possesses a $U(1)$ gauge symmetry and is invariant under the transformation:

$$\psi_a \mapsto \psi'_a = e^{i\alpha}\psi_a, \quad A_\mu \mapsto A'_\mu = A_\mu + \sqrt{N}\partial_\mu \alpha, \quad \phi \mapsto \phi' = \phi + \alpha.\tag{2.5}$$

Actually, ϕ is the fictitious Nambu-Goldstone (NG) boson field which is to be absorbed into the longitudinal component of A_μ . If we fix the gauge by the gauge transformation into the unitary gauge $\phi' = 0$ ($\alpha = -\phi$):

$$\Psi_a = \psi'_a = e^{-i\phi}\psi_a,\tag{2.6}$$

$$\tilde{A}_\mu = A'_\mu = A_\mu - \sqrt{N}\partial^\mu \phi,\tag{2.7}$$

then the Lagrangian (2.3) precisely coincides with Eq.(2.2). Thus the original Thirring model is nothing but the gauge-fixed (unitary gauge) form of our HLS model. The mass of the vector boson \tilde{A}_μ is now regarded as that of the gauge boson A_μ generated

through the Higgs mechanism [9]. This $U(1)$ case is actually identical with what is known as the Stückelberg formalism for the massive vector boson.

There are several virtues of the existence of such a gauge symmetry: First, the gauge symmetry enables us to prove straightforwardly the S-matrix unitarity through the BRS symmetry (see Ref.[26]). Secondly, actual calculations, particularly loop calculations, are generally hopeless in the unitary gauge, while the HLS provides us with the privilege to take the most appropriate gauge for our particular purpose. Let us consider a general gauge $F[A] = 0$ and introduce the gauge fixing term into Eq.(2.3):

$$\mathcal{L}_{\psi,A} = \sum_a \bar{\psi}_a i\gamma^\mu D_\mu \psi_a + \frac{1}{2G}(A_\mu - \sqrt{N}\partial_\mu \phi)^2 - \frac{1}{2}F[A] \left(\frac{1}{\xi(\partial^2)} F[A] \right), \quad (2.8)$$

where, by introducing the momentum- (derivative-) dependence of the gauge fixing parameter ξ , we have formulated at Lagrangian level the so-called nonlocal gauge[19] which has been discussed only at the SD equation level. The covariant gauge is given by $F[A] = \partial_\mu A^\mu$ in which the fictitious NG boson ϕ is not decoupled (except the Landau gauge $\xi = 0$).

More interesting gauge is the R_ξ gauge[27],

$$F[A] = \partial_\mu A^\mu + \sqrt{N} \frac{\xi(\partial^2)}{G} \phi, \quad (2.9)$$

which can again be a nonlocal gauge through the dependence of ξ on the derivative. The gauge fixing term in the nonlocal R_ξ gauge is given by

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2} \left(\partial_\mu A^\mu + \sqrt{N} \frac{\xi(\partial^2)}{G} \phi \right) \frac{1}{\xi(\partial^2)} \left(\partial_\nu A^\nu + \sqrt{N} \frac{\xi(\partial^2)}{G} \phi \right). \quad (2.10)$$

Putting Eq.(2.3) and Eq.(2.10) together, we arrive at the Lagrangian in the nonlocal R_ξ gauge:

$$\mathcal{L} = \mathcal{L}_{\psi,A} + \mathcal{L}_\phi, \quad (2.11)$$

$$\mathcal{L}_{\psi,A} = \sum_a \bar{\psi}_a i\gamma^\mu D_\mu \psi_a + \frac{1}{2G}(A_\mu)^2 - \frac{1}{2}\partial_\mu A^\mu \left(\frac{1}{\xi(\partial^2)} \partial_\nu A^\nu \right), \quad (2.12)$$

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2G}(\xi(\partial^2)\phi)\phi, \quad (2.13)$$

where we have rescaled the ϕ as $\sqrt{N/G}\phi \mapsto \phi$. Thus the fictitious NG boson ϕ is completely decoupled independently of $\xi(\partial^2)$ in the R_ξ gauge, whether ξ is nonlocal or not. It is straightforward to prove that the above Lagrangian (even in the nonlocal gauge) possesses the BRS symmetry (see Appendix). Eq.(2.12) might appear as if we added the “covariant gauge fixing term” to Eq.(2.2), although there is no gauge symmetry and \tilde{A}_μ is not a gauge field in Eq.(2.2). Such a confusion was actually made by some authors[12, 13] who happened to arrive at the Lagrangian having the same form as Eq.(2.2) in the case of constant ξ . This Lagrangian (2.12), whether the gauge parameter is nonlocal or not, can only be justified through the HLS in the R_ξ gauge. If we take the covariant gauge, on the other hand, the field ϕ does not decouple except for the Landau gauge as we have already mentioned.

In the next section we shall demonstrate that the analysis of the fermion dynamical mass generation in the ladder SD equation can be greatly simplified by taking the nonlocal gauge[19] of the R_ξ gauge for HLS. The nonlocal gauge is the gauge in which the fermion gets no wave function renormalization, i.e., $A(-p^2) = 1$ for the fermion propagator $iS^{-1}(p) = A(-p^2)\not{p} - B(-p^2)$. This gauge is necessary for the ladder (bare vertex) approximation to be consistent with the WT identity for the $U(1)$ gauge symmetry (or the current conservation of the global $U(1)$ symmetry in the original Thirring model), which actually requires no wave function renormalization. In the usual gauge ($\xi = \text{constant}$) including the Landau gauge ($\xi = 0$), on the other hand, we cannot arrange $A(-p^2) = 1$ without modifying the bare vertex into a complicated one consistent with the WT identity in rather arbitrary way.

Thirdly, the massless vector boson limit (limit of infinite four-fermion coupling constant) can be taken smoothly in the gauges other than the unitary gauge (original Thirring model) in the HLS formalism, so that our result in (2+1) dimensions can be compared with QED₃, which is impossible in the unitary gauge. Such a massless limit is also interesting in the composite models of gauge bosons[9, 17, 28].

Fourthly, the HLS can also be used to settle the regularization ambiguity on the induced CS term in (2+1)-dimensional Thirring model. Without gauge symmetry, any

regularization could be equally allowed, which then leads to contradictory result on whether or not the CS term is induced by the fermion loop. Once the HLS is explicit, regularization must be such as to preserve the HLS (Pauli-Villars regularization), which then concludes that the CS term is actually induced in the same way as in QED₃[18]. Then there exists CS term for the odd number of 2-component fermions even in the symmetric phase where the fermion mass is not generated, while for the even number of 2-component fermions it can be arranged to cancel each other within the pairs of the regulators.

At this point one might still suspect that the HLS is just a redundant degree of freedom and plays no significant role on physics, since there is no kinetic term for A_μ at tree level. However, as is well known in the D ($2 \leq D < 4$)-dimensional nonlinear sigma model like CP^{N-1} model, the HLS gauge boson acquires kinetic term through loop effects[9]. Moreover, there exist realistic examples of such dynamical gauge bosons of HLS realized in Nature: The vector mesons (ρ, ω, \dots) are successfully described as the dynamical gauge bosons of the HLS in the nonlinear chiral Lagrangian [29, 9]. We shall demonstrate in the next section that in the case at hand this phenomenon actually takes place, once the fermion gets dynamical mass from the nonperturbative loop effects in the SD equation. In passing, the limit of infinite four-fermion coupling can be taken only through the HLS, namely, the massless gauge boson can be generated dynamically only when the manifest gauge symmetry does exist [9].

Our HLS Lagrangian can easily be extended to the non-Abelian case by use of the “ $U(n)/U(n)$ ” nonlinear sigma model which is gauge equivalent to $U(n)_{\text{global}} \times U(n)_{\text{local}}$ model[9, 17]:

$$\mathcal{L} = \sum_a \bar{\psi}_a i \gamma^\mu D_\mu \psi_a - \frac{N}{G} \text{tr} \left[(D_\mu u \cdot u^\dagger)^2 \right], \quad (2.14)$$

where $A_\mu = A_\mu^\alpha T^\alpha$, and $u = e^{i\phi}$, $\phi = \phi^\alpha T^\alpha$, with T^α being the $U(n)$ generators. Actually, Eq.(2.14) is gauge equivalent to the Thirring model having the interaction

$$- \frac{G}{2N} \sum_{a,b,\alpha} (\bar{\Psi}_a \gamma^\mu T^\alpha \Psi_a) (\bar{\Psi}_b \gamma_\mu T^\alpha \Psi_b). \quad (2.15)$$

In contrast to the $U(1)$ case, however, the fictitious NG bosons ϕ in the non-Abelian

case are not decoupled even in the R_ξ gauge, which would make the SD equation analysis rather complicated.

§ 3. Schwinger-Dyson Equation

In this section we study the fermion dynamical mass generation in the D ($2 \leq D < 4$)-dimensional massless Thirring model through the SD equation. First of all, we must clarify what symmetry is to be dynamically broken by the dynamical generation of fermion mass. In this respect $D = 3$ is rather special. Since we wish to study the fermion mass generation in D ($2 \leq D < 4$) dimensions which contain $D = 3$, we here identify the types of fermion mass and the symmetries to be broken in (2+1) dimensions [30].

§ 3.1. Chiral and parity symmetries in (2 + 1) dimensions

In (2+1) dimensions the simplest representation of γ matrices is the one with respect to the 2×2 matrices, or the Pauli matrices,

$$\gamma^0 = \sigma^3, \gamma^1 = i\sigma^1, \gamma^2 = i\sigma^2. \quad (3.1)$$

The 2-component fermions are denoted by χ_a with the flavor index $a = 1, \dots, N$. The parity transformation is defined by

$$\chi_a(x) \mapsto \chi'_a(x') = e^{i\delta} \sigma^1 \chi_a(x) \quad \text{for } x' = (t, -x, y). \quad (3.2)$$

Note that the mass terms for 2-component fermion, $m\bar{\chi}_a\chi_a$, are odd under the parity symmetry. We are restricting our interest to even number of fermion species and addressing ourselves to the question whether the symmetries of the classical Lagrangian are preserved at quantum level or not. In this case it is convenient to write the theory in terms of 4-component spinors $\psi_a \equiv \begin{pmatrix} \chi_a \\ \chi_{N+a} \end{pmatrix}$ with flavor index $a = 1, \dots, N$ as we did in Section 2. The three 4×4 γ matrices can be taken to be

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \gamma^2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad (3.3)$$

and then there are three more 4×4 matrices

$$\gamma^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau \equiv -\gamma^5\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.4)$$

which, together with the identity, constitute generators of the global $U(2)$ symmetry, i.e., $\Sigma^0 = I$, $\Sigma^1 = -i\gamma^3$, $\Sigma^2 = \gamma^5$, $\Sigma^3 = \tau$. Then the (2+1)-dimensional massless Thirring model is invariant under the so-called “chiral” transformation

$$\psi_a \mapsto \psi'_a = (U\psi)_a, \quad U \equiv \exp\left(i\omega^{i\alpha}\frac{\Sigma^i}{2} \otimes T^\alpha\right), \quad (3.5)$$

where T^α denote the generators of $U(N)$, so that the full chiral symmetry of the theory is $U(2N)$ as expected. The parity transformation for 4-component fermions is composed of that of 2-component fermions and the exchange between the upper and the lower 2-component fermions, specifically

$$\psi_a(x) \mapsto \psi'_a(x') = i\gamma^3\gamma^1\psi_a(x), \quad (3.6)$$

and the corresponding operation on the gauge field becomes

$$A_\mu(x) \mapsto A'_\mu(x') = (-1)^{\delta_{\mu 1}} A_\mu(x). \quad (3.7)$$

Now we identify the peculiarity of $D = 3$ dimensions; the full global symmetries of the Lagrangian (2.1) (or equivalently Eq.(2.3)) are the parity and the global $U(2N)$ chiral symmetry. Accordingly, in (2+1)-dimensional case, order parameter of the chiral symmetry breaking $U(2N) \rightarrow U(N) \times U(N)$ is the parity-invariant mass defined by

$$m\bar{\psi}_a\psi_a = m\bar{\chi}_a\chi_a - m\bar{\chi}_{N+a}\chi_{N+a}, \quad (3.8)$$

while order parameter of the parity symmetry breaking is given by another type of mass term

$$m\bar{\psi}_a\tau\psi_a = m\bar{\chi}_a\chi_a + m\bar{\chi}_{N+a}\chi_{N+a}. \quad (3.9)$$

Though at this stage we do not yet know whether the dynamical symmetry breaking really occurs or not, we know what the breaking pattern should be once it happens,

thanks to the Vafa-Witten theorem[20]. Namely, since the tree-level gauge action corresponding to Eq.(2.3) is real and positive semi-definite in Euclidean space, energetically favorable is a parity conserving configuration consisting of half the 2-component fermions acquiring equal positive masses and the other half equal negative masses. Such a parity-conserving mass is indeed generated, as we shall show through the SD equation. It was also confirmed in QED₃ where the classical action shares the same structure as ours except for the kinetic term of the gauge field, both satisfying condition of the real positivity in Euclidean space [30]. Moreover, the parity violating pieces including the induced CS term[18] do not appear in the gauge sector whenever the number of 2-component fermions is even. According to the above arguments, the pattern of symmetry breaking we shall consider is not for the parity but for the chiral symmetry. Thus we investigate the dynamical mass of the type $m\bar{\psi}\psi$ in the SD equation. In $D(\neq 3)$ dimensions, on the other hand, such a mass breaks the chiral $U(N) \times U(N)$ symmetry of Eq.(2.2) (also Eq.(2.3)) down to a diagonal $U(N)$ symmetry. Incidentally, the $U(1)$ subgroup of this diagonal $U(N)$ was actually enlarged into the $U(1)_{\text{global}} \times U(1)_{\text{local}}$ by the HLS in Section 2. (See Ref.[9].)

§ 3.2. Schwinger-Dyson equation in the nonlocal R_ξ gauge

Taking the above arguments into account, we now study the SD equation to confirm whether the chiral symmetry is spontaneously broken or not in the D ($2 \leq D < 4$) dimensional Thirring model. We write the full fermion propagator as $S(p) = i[A(-p^2)\not{p} - B(-p^2)]^{-1}$, with B being the order parameter of the chiral symmetry which preserves the parity in (2+1) dimensions. Then the SD equation for Eq.(2.12) is written as follows:

$$(A(-p^2) - 1)\not{p} - B(-p^2) = -\frac{1}{N} \int \frac{d^D q}{i(2\pi)^D} \gamma_\mu \frac{A(-q^2)\not{q} + B(-q^2)}{A^2(-q^2)q^2 - B^2(-q^2)} \Gamma_\nu(p, q) iD^{\mu\nu}(p - q), \quad (3.10)$$

where $\Gamma_\nu(p, q)$ and $D_{\mu\nu}(p - q)$ denote the full vertex function and the full gauge boson propagator, respectively. We should apply some appropriate approximations to this equation so as to reduce it to the solvable integral equation for the mass function

$$M(-p^2) = B(-p^2)/A(-p^2).$$

Following the spirit of the analysis of QED₃[30], we here adopt an approximation based on the large N arguments, in which $\Gamma_\nu(p, q)$ and $D_{\mu\nu}(p - q)$ are those at the $1/N$ leading order, namely, the bare vertex and the one-loop vacuum polarization of massless fermion loop, respectively:

$$\Gamma_\nu(p, q) = \gamma_\nu, \quad (3.11)$$

$$-iD^{\mu\nu}(k) = d(-k^2) \left(g^{\mu\nu} - \eta(-k^2) \frac{k^\mu k^\nu}{k^2} \right), \quad (3.12)$$

$$d(-k^2) = \frac{1}{G^{-1} - \Pi(-k^2)}, \quad \eta(-k^2) = \frac{\xi(-k^2) \Pi(-k^2) - k^2}{\xi(-k^2) G^{-1} - k^2}, \quad (3.13)$$

where we have adopted a nonlocal R_ξ gauge, Eq.(2.10), with the momentum-dependent gauge parameter $\xi(-k^2)$, and $\Pi(-k^2)$ is the one-loop vacuum polarization of massless fermions:

$$\Pi^{\mu\nu}(k) = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \Pi(-k^2), \quad (3.14)$$

which is readily calculated to be:

$$\Pi(-k^2) = -\frac{2 \operatorname{tr} I}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) B\left(\frac{D}{2}, \frac{D}{2}\right) (-k^2)^{D/2-1}, \quad (3.15)$$

with $\operatorname{tr} I$ being the trace of unit matrix in spinor indices (4 for $2 < D < 4$ and 2 for $D = 2$).

Then the SD equation (3.10) is reduced to the following coupled equations for $A(-p^2)$ and $B(-p^2)$:

$$A(-p^2) - 1 = \frac{1}{Np^2} \int \frac{d^D q}{i(2\pi)^D} \frac{A(-q^2)}{A^2(-q^2)q^2 - B^2(-q^2)} d(-k^2) \left[\{ \eta(-k^2) + 2 - D \} (p \cdot q) - \frac{2(k \cdot p)(k \cdot q)}{k^2} \eta(-k^2) \right], \quad (3.16)$$

$$B(-p^2) = -\frac{1}{N} \int \frac{d^D q}{i(2\pi)^D} \frac{B(-q^2)}{A^2(-q^2)q^2 - B^2(-q^2)} d(-k^2) [D - \eta(-k^2)], \quad (3.17)$$

where $k_\mu = p_\mu - q_\mu$. These are our basic equations.

At first sight Eqs.(3.16), (3.17) might seem to be trivial in $1/N$ expansion, since L.H.S. is formally of $O(1/N)$ whereas R.H.S. is of $O(1)$, which then would imply a trivial

solution, $A(-p^2) = 1$ and $B(-p^2) = 0$, at the $1/N$ leading order. However, as was realized in QED₃[30], these equations are self-consistent nonlinear equations through which $A(-p^2)$ and $B(-p^2)$ may arrange themselves to balance the N -dependence of L.H.S. and that of R.H.S. in a nontrivial manner. In fact, N -dependence of the solution might be non-analytic in N as was the case in QED₃[21]. We wish to find such a nonperturbative nontrivial solution by just examining the Eqs.(3.16), (3.17) for finite N .

A technical issue to solve Eqs.(3.16), (3.17) is how to handle the coupled SD equations Eqs.(3.16), (3.17). Here we follow the nonlocal gauge proposed by Georgi et al.[19, 23] which reduces the coupled SD equations into a single equation for $B(-p^2)$ by requiring $A(-p^2) \equiv 1$ in Eq.(3.16) by use of the freedom of gauge choice. This is actually the gauge in which the bare vertex approximation can be consistent with the WT identity for HLS (or the current conservation), i.e., $A(0) = 1$. This is in sharp contrast to the ordinary (momentum-independent) covariant gauge or even the Landau gauge with $\eta(-k^2) = 1$ ($\xi(-k^2) = 0$), in which the bare vertex approximation is not consistent with the WT identity. In the nonlocal gauge $B(-p^2)$ itself is a mass function, i.e., $M(-p^2) = B(-p^2)$.

Requiring $A(-p^2) \equiv 1$ in the nonlocal gauge, we perform the angular integration of Eqs.(3.16), (3.17) in Euclidean space (hereafter in this section we use the Euclidean notation):

$$0 = \int_0^\pi d\theta \sin^D \theta \left[\frac{1}{D-1} \frac{d}{dk^2} \left\{ d(k^2)(\eta(k^2) + D - 2) \right\} + \frac{\eta(k^2)d(k^2)}{k^2} \right], \quad (3.18)$$

$$B(p^2) = \frac{1}{N} \int_0^{\Lambda^{D-2}} d(q^{D-2}) K(p, q; G) \frac{q^2 B(q^2)}{q^2 + B^2(q^2)}, \quad (3.19)$$

where we have introduced ultraviolet (UV) momentum cutoff Λ and the kernel $K(p, q; G)$ is given by

$$K(p, q; G) = \frac{1}{(D-2)2^{D-1}\pi^{(D+1)/2}\Gamma(\frac{D-1}{2})} \int_0^\pi d\theta \sin^{D-2} \theta d(k^2)[D - \eta(k^2)], \quad (3.20)$$

with $k^2 = p^2 + q^2 - 2pq \cos \theta$. It is easily seen that Eq.(3.18) is used to determine the

gauge fixing function $\eta(k^2)$:

$$\eta(k^2) = (D-2) \left[\frac{D-1}{k^{2(D-1)} d(k^2)} \int_0^{k^2} d\zeta \zeta^{D-2} d(\zeta) - 1 \right]. \quad (3.21)$$

Substituting $d(k^2)$ in Eq.(3.13) into the above relation, we determine $\eta(k^2)$:

$$\begin{aligned} \eta(k^2) &= (D-2) \left[\left(1 + \frac{Gk^{D-2}}{C_D} \right) {}_2F_1 \left(1, 1 + \frac{D}{D-2}, 2 + \frac{D}{D-2}; -\frac{Gk^{D-2}}{C_D} \right) - 1 \right], \\ C_D^{-1} &\equiv \frac{2 \operatorname{tr} I}{(4\pi)^{D/2}} \Gamma \left(2 - \frac{D}{2} \right) B \left(\frac{D}{2}, \frac{D}{2} \right), \end{aligned} \quad (3.22)$$

with ${}_2F_1(a, b, c; z)$ being the hypergeometric function. Eq.(3.22) actually forces the gauge fixing parameter ξ to be a function of k^2 . Once we determined $\eta(k^2)$ and hence the kernel Eq.(3.20), our task is now reduced to solving a single SD equation for the mass function $B(p^2)$, Eq.(3.19), which is much more tractable.

The kernel $K(p, q; G)$ in Eq.(3.20) is a positive function for positive arguments p , q , and G , since $D - \eta(k^2)$ is positive for positive k^2 . Moreover, the kernel depends on the arguments p and q only through $k^2 = p^2 + q^2 - 2pq \cos \theta$, so that it has the symmetry under the exchange of p and q . These kinematical properties of the kernel $K(p, q; G)$ are essential to proving the existence of a nontrivial solution of the SD equation (3.19) in the next subsection.

§ 3.3. Existence of nontrivial solution and critical line

Now we investigate existence of the nontrivial solution for the SD equation (3.19) for $2 \leq D < 4$, based on the method of Refs.[31, 32]. In $D = 2$ dimensions the SD equation reduces to the gap equation for the constant dynamical fermion mass as in the Gross-Neveu model. We can explicitly solve this full gap equation. For $2 < D < 4$ we shall use the bifurcation method[32] to solve the SD equation.

Let us first consider $2 < D < 4$. The integral equation (3.19) always has a trivial solution $B(p^2) \equiv 0$. We are interested in the vicinity of the phase transition point where the nontrivial solution also starts to exist. Such a bifurcation point is identified by the existence of an infinitesimal solution $\delta B(p^2)$ around the trivial solution $B(p^2) \equiv 0$ [32].

Then we obtain the linearized equation for $\delta B(p^2)$:

$$\delta B(p^2) = \frac{1}{N} \int_{m^{D-2}}^{\Lambda^{D-2}} d(q^{D-2}) K(p, q; G) \delta B(q^2), \quad (3.23)$$

where we introduced the IR cutoff m . It is enough for us to show the existence of a nontrivial solution of the bifurcation equation (3.23) [31]. Particularly, we can obtain the exact phase transition point where the bifurcation takes place. Since we normalize the solution as $m = \delta B(m^2)$, m is nothing but the dynamically generated fermion mass.

Rescaling $p = \Lambda x^{\frac{1}{D-2}}$ and $\delta B(p^2) = \Lambda \Sigma(x)$, we rewrite Eq.(3.23) as follows:

$$\Sigma(x) = \frac{1}{N} \int_{\sigma_m}^1 dy \tilde{K}(x, y; g) \Sigma(y), \quad (3.24)$$

where we introduced $\sigma_m = (m/\Lambda)^{D-2}$ ($0 < \sigma_m \leq 1$), the dimensionless four-fermion coupling constant $g = G/\Lambda^{2-D}$ and

$$\tilde{K}(x, y; g) \equiv K(x^{\frac{1}{D-2}}, y^{\frac{1}{D-2}}; g). \quad (3.25)$$

As we mentioned in the end of the last subsection, the kernel $\tilde{K}(x, y; g)$ is positive and symmetric:

$$\tilde{K}(x, y; g) = \tilde{K}(y, x; g) > 0, \quad \text{for } x, y \text{ and } g \geq 0. \quad (3.26)$$

This is the most important property for the existence proof of the nontrivial solution [31].

Let us consider the linear integral equation:

$$\phi(x) = \frac{1}{\lambda} \int_{\sigma_m}^1 dy \tilde{K}(x, y; g) \phi(y), \quad (3.27)$$

whose eigenvalues and eigenfunctions are denoted by $\lambda_n(g, \sigma_m)$ ($|\lambda_n| \geq |\lambda_{n+1}|$; $n = 1, 2, \dots$) and $\phi_n(x)$, respectively. The kernel $\tilde{K}(x, y; g)$ is a symmetric one and hence satisfies the following property:

$$\sum_{n=1}^{\infty} \lambda_n^2(g, \sigma_m) = \int_{\sigma_m}^1 \int_{\sigma_m}^1 dx dy [\tilde{K}(x, y; g)]^2 < \infty. \quad (3.28)$$

The R.H.S. of Eq.(3.28) gives the upper bound for each eigenvalue $\lambda_n(g, \sigma_m)$. Furthermore, using the positivity of the symmetric kernel (see Eq.(3.26)), we can prove that the

maximal eigenvalue $\lambda_1(g, \sigma_m)$ is always positive and the corresponding eigenfunction $\phi_1(x)$ has a definite sign (nodeless solution).

In the bifurcation equation (3.24) this implies the following: *If N is equal to the maximal eigenvalue of the kernel $\lambda_1(\alpha, \sigma_m)$: $N = \lambda_1(\alpha, \sigma_m)$, then there exists a nontrivial nodeless solution $\Sigma(x) = \phi_1(x)$ besides a trivial one.* $N = \lambda_1(g, \sigma_m)$ determines a line on (N, g) plane which is specified by σ_m . Hence the above statement means each line with the parameter σ_m corresponds to the dynamically generated mass $m = \Lambda(\sigma_m)^{\frac{1}{D-2}}$. Now we introduce $N_{\text{cr}}(g)$ defined by

$$N_{\text{cr}}(g) = \lambda_1(g, \sigma_m \rightarrow 0). \quad (3.29)$$

As σ_m approaches zero, the corresponding line also approaches the critical line on (N, g) plane; $N = N_{\text{cr}}(g)$. Moreover, since $\lambda_1(g, \sigma_m)$ is the maximal eigenvalue of the kernel, there is no non-zero solution for N larger than $\lambda_1(g, \sigma_m)$. Through these consideration we can conclude that *if the inequality $N < N_{\text{cr}}(g)$ is satisfied, then the fermion mass is dynamically generated.* Existence of the critical line, $N = N_{\text{cr}}(g)$ or $g = g_{\text{cr}}(N)$, in the two-parameter space is somewhat analogous to that in the gauged NJL model[33].

Although it is difficult to obtain the explicit form of the critical line $N = N_{\text{cr}}(g)$ in the general case, we can do it in the limit of infinite four-fermi coupling constant, $g \rightarrow \infty$. Let us discuss the (2+1) dimensions for definiteness, in which case the kernel reads

$$K(x, y; g \rightarrow \infty) = \frac{32}{3\pi^2} \min \left\{ \frac{1}{x}, \frac{1}{y} \right\}. \quad (3.30)$$

Then the bifurcation equation (3.23) in (2+1) dimensions is rewritten into a differential equation

$$\frac{d}{dx} \left(x^2 \frac{d\Sigma(x)}{dx} \right) = -\frac{32}{3\pi^2 N} \Sigma(x), \quad (3.31)$$

plus boundary conditions

$$\Sigma'(\sigma_m) = 0, \quad (\text{IR B.C.}) \quad (3.32)$$

$$[x\Sigma'(x) + \Sigma(x)]_{x=1} = 0. \quad (\text{UV B.C.}) \quad (3.33)$$

Eqs.(3.31), (3.32) and (3.33) are the same as those in QED₃ [30, 22, 23]. When $N > N_{\text{cr}} \equiv 128/3\pi^2$, there is no nontrivial solution of Eq.(3.31) satisfying the boundary

conditions, while for $N < N_{\text{cr}}$ the following bifurcation solutions exist:

$$\begin{aligned}\Sigma(x) &= \frac{\sigma_m}{\sin(\frac{\omega}{2}\delta)} \left(\frac{x}{\sigma_m}\right)^{-\frac{1}{2}} \sin \left\{ \frac{\omega}{2} \left[\ln \frac{x}{\sigma_m} + \delta \right] \right\}, \\ \omega &\equiv \sqrt{N_{\text{cr}}/N - 1}, \quad \delta \equiv 2\omega^{-1} \arctan \omega,\end{aligned}\tag{3.34}$$

where σ_m is given by the UV boundary condition (3.33):

$$\frac{\omega}{2} \left[\ln \frac{1}{\sigma_m} + 2\delta \right] = n\pi, \quad n = 1, 2, \dots \tag{3.35}$$

The solution with $n = 1$ is the nodeless (ground state) solution whose scaling behavior is read from Eq.(3.35):

$$\frac{m}{\Lambda} = e^{2\delta} \exp \left[-\frac{2\pi}{\sqrt{N_{\text{cr}}/N - 1}} \right]. \tag{3.36}$$

The critical number $N_{\text{cr}} = 128/3\pi^2$ is equivalent to the one in QED₃ with the nonlocal gauge fixing[22, 23].

Here we discuss the reason why our bifurcation equation at $g \rightarrow \infty$ in (2+1) dimensions is the same as the one in QED₃. In the nonlocal gauge the SD equation for QED₃ reads[23]

$$\delta B(p^2) = \frac{1}{N} \int_0^\alpha dq K_{\text{QED}}(p, q; \alpha) \delta B(q^2), \tag{3.37}$$

$$K_{\text{QED}}(p, q; \alpha) = \frac{\alpha}{4\pi^2} \int_0^\pi d\theta \sin \theta d_{\text{QED}}(k^2) [3 - \eta(k^2)], \tag{3.38}$$

where the scale α is defined by $\alpha = Ne^2$ with the gauge coupling e and the nonlocal gauge function $\eta(k^2)$ is given by Eq.(3.21) with $d(k^2)$ replaced by $\alpha d_{\text{QED}}(k^2)$. It is known that contribution to the kernel comes mainly from the momentum region $k < \alpha$ [30]. Noting that $\Pi(k^2) \sim k$ from Eq.(3.15), we can expand $\alpha d_{\text{QED}}(k^2)$ in k/α :

$$\alpha d_{\text{QED}}(k^2) = \frac{\alpha}{k^2 - \alpha \Pi(k^2)} = \frac{-1}{\Pi(k^2)} \left\{ 1 + \mathcal{O} \left(\frac{k}{\alpha} \right) \right\}. \tag{3.39}$$

On the other hand, $d(k^2)$ in our case becomes identical with the first term of Eq.(3.39) in the limit of $g \rightarrow \infty$:

$$d(k^2) = \frac{1}{\Lambda g^{-1} - \Pi(k^2)} \rightarrow \frac{-1}{\Pi(k^2)}. \tag{3.40}$$

In spite of the big difference in the general form, $\alpha d_{\text{QED}}(k^2)$ at $k/\alpha \ll 1$ and $d(k^2)$ at $g^{-1} \ll 1$ are both dominated by the same vacuum polarization $\Pi(k^2)$, which yields the same $\eta(k^2)$ and hence the same kernel. This is the reason for the coincidence of the value of N_{cr} with that of the QED₃.

However, we should note an essential difference of our case from the QED₃. Since the asymptotic behavior of the HLS gauge boson propagator is $\sim 1/k$, the loop integration appearing in R.H.S. of the SD equation (3.10) is logarithmically divergent. This is due to lack of the kinetic term of the HLS gauge boson at tree level, in contrast to the photon in QED₃ whose asymptotic behavior is $\sim 1/k^2$. Hence, in order to keep the integral in Eq.(3.10) to be finite, we must introduce the cutoff Λ , whereas in QED₃ the gauge coupling constant α provides an intrinsic mass scale which plays a role of the natural cutoff [30]. Therefore, different from QED₃, Λ should be removed by taking the limit $\Lambda \rightarrow \infty$ in such a way as to keep the physical quantity like the fermion dynamical mass to be finite. This procedure corresponds to the renormalization à la Miransky proposed in the strong coupling QED₄[34]. This renormalization defines the continuum theory at the UV fixed point located at the critical line $N = N_{\text{cr}}(g)$ or $g = g_{\text{cr}}(N)$ in much the same way as the gauged NJL model[6].

Next we discuss the $D = 2$ case, in which $\eta(k^2) = 0$ from Eq.(3.21) and thereby the kernel does not depend on p . Then the SD equation (3.17) is written as

$$B(p^2) = \frac{1}{N(1 + \pi/G)} \int_0^\Lambda dq \frac{qB(q^2)}{q^2 + B^2(q^2)}. \quad (3.41)$$

The R.H.S. of Eq.(3.41) has no p dependence, so that the fermion mass function is just a constant mass $B(p^2) \equiv m$. This salient feature can only be realized under the nonlocal gauge $\xi(k^2) \neq \text{const.}$ we have chosen. Therefore, as in the Gross-Neveu model, the above equation gives the gap equation:

$$m = \frac{1}{N(1 + \pi/G)} \int_0^\Lambda dq \frac{qm}{q^2 + m^2}, \quad (3.42)$$

which has a nontrivial solution $m \neq 0$ for arbitrary N when $G > 0$ or $G < -\pi$:

$$\ln \left(1 + \frac{\Lambda^2}{m^2} \right) = 2N \left(1 + \frac{\pi}{G} \right). \quad (3.43)$$

Now the (N, G) plane is divided into three regions with $N > 0$; 1. $G > 0$, 2. $-\pi < G < 0$, and 3. $G < -\pi$. When we take the continuum limit $m/\Lambda \rightarrow 0$ as in the case of $2 < D < 4$, we reach the critical line $G = 0$ and $N > 0$ which may be interpreted as a trivial UV fixed line. On the other hand, $G = -\pi$ is only reached by $\Lambda/m \rightarrow 0$ (maybe a nontrivial “IR fixed line”) and has nothing to do with the continuum limit. The β -function in the broken phase ($G > 0$, $G < -\pi$) is given by

$$\beta_N(G) \equiv \Lambda \left. \frac{\partial G(\Lambda)}{\partial \Lambda} \right|_N = -\frac{G^2}{\pi N} \left[1 - e^{-2N(1+\frac{\pi}{G})} \right]. \quad (3.44)$$

The region 2 allows only the trivial solution $m = 0$. In the next section we shall derive the bosonization in (1+1) dimensions through the dual transformation for HLS and show that the theory with $-\pi < G < 0$ lies not in the symmetric phase but has no ground state so that the trivial solution $m = 0$ depicts an unstable extremum. It means that the theory only has a broken phase with $G > 0$ or $G < -\pi$. Therefore the results based on the SD equation perfectly coincide with the exact results obtained by operator methods[10] and those in Section 5. This agreement is quite encouraging for the reliability of the SD equation in D ($2 < D < 4$) dimensions as well as $D = 2$.

§ 4. Dynamical Gauge Boson

In this section we discuss the dynamical pole generation of the HLS gauge boson which is merely the auxiliary field at tree level. It is obvious that in the chiral symmetric phase where the fermions remain massless, there is no chance for the HLS gauge boson to develop a pole due to fermion loop effect, since a massive vector bound state, if it is formed, should decay into massless fermion pair immediately. In fact the gauge boson propagator in Eq.(3.12) and Eq.(3.13) with the contribution from the vacuum polarization of massless fermions in Eq.(3.14) has no pole in the time-like momentum region. However, once the fermion acquires the mass, the HLS gauge boson propagator can have a pole structure due to the massive fermion loop effect[9]. We here discuss the vacuum polarization tensor of massive fermion loop in D ($2 \leq D < 4$) dimensions.

At this point one might suspect that the fermion mass effect on the vacuum polarization tensor may affect the analysis of the SD equation in Section 3 where the vacuum

polarization tensor (3.14) was calculated by the massless fermion loop. However, the fermion mass effect on the SD equation through the vacuum polarization tensor enters the kernel only as a linear or higher terms in $\delta B(-p^2)$ in the bifurcation form of Eq.(3.23). There exists a linear term of $\delta B(-p^2)$ already in the integral of Eq.(3.23), so that the mass effect on the vacuum polarization tensor yields only higher order in $\delta B(-p^2)$ and can be neglected in the bifurcation equation. Thus our analysis in Section 3 is totally unaffected by inclusion of the dynamically generated fermion mass in the vacuum polarization tensor.

Suppose that the fermion acquire the dynamical mass $m = \delta B(m^2)$. Disregarding the momentum-dependence of the mass function for simplicity, we can calculate the one-loop vacuum polarization tensor for the HLS gauge boson as

$$\Pi^{\mu\nu}(k) = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \Pi(-k^2), \quad (4.1)$$

where

$$\begin{aligned} \Pi(-k^2) &= \frac{2 \operatorname{tr} I}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx \frac{x(1-x)k^2}{[m^2 - x(1-x)k^2]^{2-D/2}} \\ &= \frac{\operatorname{tr} I \Gamma\left(2 - \frac{D}{2}\right)}{3(4\pi)^{D/2}} \frac{k^2}{m^{4-D}} {}_2F_1\left(2, 2 - \frac{D}{2}, \frac{5}{2}; \frac{k^2}{4m^2}\right). \end{aligned} \quad (4.2)$$

Since the function $d(-k^2)$ in Eq.(3.12) is defined by $d(-k^2) = [G^{-1} - \Pi(-k^2)]^{-1}$, the pole mass M_V of the dynamical gauge boson, if it exists, is defined by the following equation:

$$G^{-1} = \Pi(-M_V^2), \quad 0 \leq M_V^2 < 4m^2. \quad (4.3)$$

First of all we observe that $M_V \rightarrow 0$ as $G \rightarrow \infty$ for arbitrary D . This limit is well-defined only through the introduction of HLS which becomes a spontaneously unbroken gauge symmetry. On the other hand, the original Thirring model corresponding to the unitary gauge of HLS becomes ill-defined in this limit.

It is easily found that when k^2 approaches $4m^2$, $\Pi(-k^2)$ diverges in $2 \leq D \leq 3$. Thus the solution of Eq.(4.3) exists in $2 \leq D \leq 3$ for any magnitude of coupling constant G (although it should be stronger than the critical value over which the dynamical fermion

mass is generated). Once the fermion acquires the mass m dynamically, the HLS gauge boson always develops a pole at $M_V < 2m$. Specifically in (1+1) dimensions the mass function is really momentum-independent, $B(-p^2) \equiv m$, and the above calculation becomes exact. Then the HLS gauge boson has a pole in the broken phase with $G > 0$:

$$\frac{1}{\pi} \left[\frac{4m^2}{M_V \sqrt{4m^2 - M_V^2}} \tan^{-1} \sqrt{\frac{M_V^2}{4m^2 - M_V^2}} - 1 \right] = G^{-1}. \quad (4.4)$$

In (2+1) dimensions the HLS gauge boson pole is given by

$$\frac{2}{3\pi} \left[\frac{4m^2 + M_V^2}{2M_V} \tanh^{-1} \frac{M_V}{2m} - m \right] = G^{-1}. \quad (4.5)$$

In the case of $3 < D < 4$, on the other hand, the R.H.S. of Eq.(4.3) remains finite even when $k^2 \rightarrow 4m^2$, then there exists a lower bound of G under which the HLS gauge boson propagator has no pole. This lower bound of G is determined by

$$G_V^{-1} \equiv \Pi(-4m^2) = \frac{4trI \Gamma(2 - D/2)}{3(4\pi)^{(D/2)}} {}_2F_1(2, 2 - \frac{D}{2}, \frac{5}{2}; 1) \times m^{D-2}. \quad (4.6)$$

The intriguing feature of our case is that both G_V and m are related to a single coupling constant G . This is in contrast to the massive Thirring model where the fermion mass is given by hand, and also to the mixed model of Gross-Neveu and Thirring model where the fermion dynamical mass and the HLS gauge boson mass are separately determined by the Gross-Neveu coupling and the Thirring coupling, respectively.

§ 5. Dual Transformation and Bosonization

Now that we have reformulated the Thirring model as a gauge theory, we can further gain an insight into the theory by using a technique inherent to the gauge theory, namely, the dual transformation [25]. Let us rewrite the theory with HLS in Eq.(2.3) by use of the dual transformation, which in (1+1) dimensions leads to the bosonization of Thirring model in the context of path integral.

We first consider the path integral for the Lagrangian (2.3),

$$Z_{\text{HLS}} = \int [dA_\mu][d\phi][d\bar{\psi}_a][d\psi_a] \exp i \int d^Dx \left\{ \sum_a \bar{\psi}_a i \gamma^\mu D_\mu \psi_a + \frac{1}{2G} (A_\mu - \sqrt{N} \partial_\mu \phi)^2 \right\}. \quad (5.1)$$

Linearizing the “mass term” of gauge field by introducing an auxiliary field C_μ , we obtain a delta functional for $\partial_\mu C^\mu$ through an integration over the scalar field ϕ as follows:¹

$$\begin{aligned} & \int [d\phi] \exp i \int d^D x \frac{1}{2G} (A_\mu - \sqrt{N} \partial_\mu \phi)^2 \\ &= \int [d\phi] [dC_\mu] \exp i \int d^D x \left\{ -\frac{1}{2} C_\mu C^\mu + \frac{1}{\sqrt{G}} C_\mu (A^\mu - \sqrt{N} \partial^\mu \phi) \right\} \end{aligned} \quad (5.2)$$

$$= \int [dC_\mu] \delta(\partial_\mu C^\mu) \exp i \int d^D x \left\{ -\frac{1}{2} C_\mu C^\mu + \frac{1}{\sqrt{G}} C_\mu A^\mu \right\}, \quad (5.3)$$

where the auxiliary field C_μ at tree level is nothing but the conserved current. If we pick up C_μ by use of dual antisymmetric-tensor field $H_{\mu_1 \dots \mu_{D-2}}$ of rank $D-2$, which satisfies the Bianchi identity, we have the following relation

$$\int [dC_\mu] \delta(\partial_\mu C^\mu) \dots = \int [dC_\mu] [dH_{\mu_1 \dots \mu_{D-2}}] \delta(C_{\mu_1} - \epsilon_{\mu_1 \dots \mu_D} \partial^{\mu_2} H^{\mu_3 \dots \mu_D}) \dots \quad (5.4)$$

Substituting the above relation into the path integral and integrating out the auxiliary field C_μ , then we have

$$\begin{aligned} Z_{\text{Dual}} &= \int [dH_{\mu_1 \dots \mu_{D-2}}] [dA_\mu] [d\bar{\psi}_a] [d\psi_a] \exp i \int d^D x \left\{ \sum_a \bar{\psi}_a i \gamma^\mu D_\mu \psi_a \right. \\ &\quad \left. + \frac{(-1)^D}{2(D-1)} H_{\mu_1 \dots \mu_{D-1}} H^{\mu_1 \dots \mu_{D-1}} + \frac{1}{\sqrt{G}} \epsilon^{\mu_1 \dots \mu_D} A_{\mu_1} \partial_{\mu_2} H_{\mu_3 \dots \mu_D} \right\} \end{aligned} \quad (5.5)$$

$$\begin{aligned} &= \int [dH_{\mu_1 \dots \mu_{D-2}}] [d\bar{\psi}_a] [d\psi_a] \delta \left(\sum_a \frac{1}{\sqrt{N}} \bar{\psi}_a \gamma^{\mu_1} \psi_a + \frac{1}{\sqrt{G}} \epsilon^{\mu_1 \dots \mu_D} H_{\mu_2 \dots \mu_D} \right) \\ &\quad \exp i \int d^D x \left\{ \sum_a \bar{\psi}_a i \gamma^\mu \partial_\mu \psi_a + \frac{(-1)^D}{2(D-1)} H_{\mu_1 \dots \mu_{D-1}} H^{\mu_1 \dots \mu_{D-1}} \right\}, \end{aligned} \quad (5.6)$$

where $H_{\mu_1 \dots \mu_{D-1}} = \partial_{\mu_1} H_{\mu_2 \dots \mu_{D-1}} - \partial_{\mu_2} H_{\mu_1 \mu_3 \dots \mu_{D-1}} + \dots + (-1)^D \partial_{\mu_{D-1}} H_{\mu_1 \dots \mu_{D-2}}$. The Lagrangian (5.6) describes N “free” fermions and a “free” antisymmetric tensor field of rank $D-2$ which are, however, constrained through the delta functional. This implies that the dual field is actually a composite of the fermions. In (2+1) dimensions

¹The scalar phase ϕ can in fact be divided into two parts: $\phi = \Theta + \eta$, where Θ expressed by multi-valued function describes the topologically nontrivial sector, e.g., the creation and annihilation of topological solitons, and η given by single-valued function depicts the fluctuation around a given topological sector. Inclusion of the topological sector Θ induces a topological interaction term [25], though we neglect Θ contribution in this section, since we are interested in ϕ as the NG mode.

the dual gauge field H_μ has the vector structure as A_μ does, while in (3+1) dimensions it is the second-rank anti-symmetric tensor field $H_{\mu\nu}$, i.e., $H_{\mu\nu\rho} \sim \epsilon_{\mu\nu\rho\sigma} \bar{\psi} \gamma^\sigma \psi$.

Although the delta functional in Eq.(5.6) tells us that the dual field is a composite of the fermions, it is difficult to read directly phase structure of the Thirring model in this formalism. If we look at the tree level Lagrangian (5.6) in (2+1) dimensions, it might seem that the dual gauge field H_μ is massless independently of the phase structure. In order to understand the pole structure of the dual gauge field H_μ , however, we have to take into account the quantum effect. For that purpose we may ignore the contributions from the fermion one-loop diagrams except for the vacuum polarization, since they generate only the self-interaction terms of the gauge field A_μ or equivalently those of the dual gauge field H_μ . Therefore we compute the vacuum polarization in Eq.(5.5):

$$Z_{\text{Dual}} \approx \int [dH_\mu][dA_\mu] \exp i \int d^3x \left\{ -\frac{1}{4} H_{\mu\nu} H^{\mu\nu} + \frac{1}{\sqrt{G}} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu H_\rho - \frac{1}{2} A_\mu \left(g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\partial^2} \right) \Pi(\partial^2) A_\nu \right\}. \quad (5.7)$$

Integrating out the HLS gauge field A_μ , we obtain an effective Lagrangian for the dual gauge field H_μ without interaction terms:

$$\mathcal{L}_H = \frac{1}{2G} H_\mu \left(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) [G - \Pi^{-1}(\partial^2)] H_\nu. \quad (5.8)$$

If we compare the pole structure of H_μ with that of A_μ in Eq.(4.3), we easily find that the dual gauge field H_μ shares exactly the same pole structure with the gauge field A_μ irrespectively of the phase.

In (1+1) dimensions the relation in the delta functional in Eq.(5.6) implies nothing but the bosonization of Thirring model in the scheme of path integral, i.e., $\frac{1}{\sqrt{G}} \epsilon^{\mu\nu} \partial_\nu H \approx \frac{1}{\sqrt{N}} \bar{\psi}_a \gamma^\mu \psi_a$. Integrating the fermions in Eq.(5.5), we obtain an effective theory which consists of a pseudoscalar and a vector gauge fields:

$$Z_{2D} = \int [dH][dA_\mu] (\det i\mathcal{D})^N \exp i \int d^2x \left\{ \frac{1}{2} \partial^\mu H \partial_\mu H + \frac{1}{2\sqrt{G}} H \epsilon_{\mu\nu} F^{\mu\nu} \right\}. \quad (5.9)$$

The second term of the action in Eq.(5.9) is the (1+1) dimensional analogue of axion term which is the interaction term between the scalar and the gauge fields and takes

the form $H\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ in (3+1) dimensions. Though the computation of fermionic determinant with regularization generates the Abelian chiral anomaly, this problem is resolved by the constant shift of scalar field H in axion term. Since the fermionic determinant is computed in an exact form, i.e.,

$$-iN \ln \frac{\det i\mathcal{D}}{\det i\mathcal{D}} = \frac{1}{2\pi} \int d^2x A^\mu (g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}) A^\nu,$$

the integration over A_μ gives a free massless scalar theory as the bosonized Thirring model

$$Z_{\text{boson}} = \int [dH] \exp i \int d^2x \frac{1}{2} (1 + \frac{\pi}{G}) \partial_\mu H \partial^\mu H. \quad (5.10)$$

If G is in the region $-\pi < G < 0$, the energy per unit volume is unbounded below and hence the (1+1)-dimensional Thirring model with coupling constant G ($G > 0$ or $G < -\pi$) has only the broken phase via the fermion dynamical mass generation as in the (1+1)-dimensional Gross-Neveu model.

§ 6. Conclusion and Discussions

In this paper we have studied the Thirring model in D ($2 \leq D < 4$) dimensions and proposed how to understand it as a gauge theory through the introduction of hidden local symmetry. The advantage of manifest gauge symmetry was to let the various nonperturbative approaches tractable and provide the consistent method to treat such problems.

In the case of $2N$ 2-component fermions (or equivalently N 4-component Dirac fermions) we studied the dynamical symmetry breaking in the context of $1/N$ expansion. Since we had the manifest $U(1)$ gauge symmetry, we took a privilege to choose a nonlocal R_ξ gauge, which greatly simplified the analysis of the SD equation.

By using the bifurcation technique, we found a second order phase transition at a certain number of N and g , thus having established the existence of the critical line on the (N, g) plane. We also proved existence of the nontrivial solution rigorously. The HLS gauge boson became massless in the $g \rightarrow \infty$ limit, where the SD equation was solved analytically in (2+1) dimensions, yielding $N_{\text{cr}} = \frac{128}{3\pi^2}$ in perfect agreement

with N_{cr} in QED₃. This limit makes sense thanks to the HLS, in sharp contrast to the original Thirring model where this limit is ill-defined.

In (1+1) dimensions, on the other hand, fermion mass is always generated, no matter what value $N(> 0)$ and $G = g(> 0)$ might take: The theory is in one phase (broken phase) for $G > 0$ as in the (1+1)-dimensional Gross-Neveu model. Our result is consistent with the exact solution of the (1+1)-dimensional Thirring model. In this case there is no regularization ambiguity, since the regularization must respect the HLS as in the massless Schwinger model.

The dynamical symmetry breaking in (2+1)-dimensional Thirring model with many flavors have previously been discussed in $1/N$ expansion[12, 13, 14, 15, 16]. Here we compare our results with those of the previous authors. When the auxiliary vector field \tilde{A}_μ has been used, the authors in Ref.[12, 13] pretended it as a gauge field and added the “gauge fixing” term, though the model carries no gauge symmetry. The hidden local symmetry we found in such model explains that the gauge fixing they chose is neither the “over-gauge fixing” nor the Landau gauge but the R_ξ gauge, so that the awkward procedure of “the gauge fixing without gauge symmetry” is justified by the discovery of hidden local $U(1)$ symmetry. The ladder approximation for the vertex leads to $A(-p^2) = 1$ to be consistent with the WT identity, while it is not allowed under the Landau gauge. It can only be realized through the nonlocal R_ξ gauge we chose.

The dynamical gauge boson is generated when fermions get mass. The result in $2 \leq D \leq 3$ has a novel feature. The gauge boson pole is always developed independently of the coupling G , once the fermion acquires mass at $G > g_{\text{cr}}/\Lambda^{D-2}$. For $3 < D < 4$, on the other hand, the gauge boson pole can be generated only for $G > G_V$ which may or may not be satisfied by the coupling larger than the critical coupling $g_{\text{cr}}/\Lambda^{D-2}$. It would be interesting to see the precise relation between the critical coupling for the dynamical gauge boson generation and that of the fermion dynamical mass generation in this case.

We rewrote Thirring model with hidden local symmetry in terms of dual field. In (1+1) dimensions, we demonstrated that this dual transformation based on the intro-

duction of hidden local symmetry is a straightforward way to arrive at the bosonization of Thirring model. In (2+1) dimensions, it was also shown that both the HLS gauge field A_μ and dual gauge field H_μ share the same mass spectrum; they are massless in symmetric phase and they have equal mass in broken phase. This formulation might be useful also in D ($2 < D < 4$) dimensions.

In (2+1) dimensions we assumed two well-known results such that, if the number of 2-component fermions is even and the classical Lagrangian is parity-even, the parity-violating sector is not induced both in the effective action for the gauge field [18] and in the pattern of dynamically generated fermion masses [20]. Both of them are consequences of exact calculations and, of course, it is consistent with our results. However, the previous papers[12, 14, 15, 16] claimed appearance of parity-violating piece through quantum effect, which is opposed to the results of both Refs.[18, 20] and ours given in Section 3.

The classical action in Eq.(2.3) for odd number of massless fermions is invariant under the $U(1)$ HLS gauge transformation and the parity in (2+1) dimensions, while in the quantized theory both of them cannot be preserved simultaneously. Since the regularization is to be specified so as to keep the $U(1)$ HLS (Pauli-Villars regularization), there is no regularization ambiguity in our case and hence the parity-violating anomaly arises as in QED₃ [18]. Therefore, the gauge theory described by the effective gauge field action lies in Chern-Simons Higgs phase:

$$\mathcal{L}_{\text{eff}}(A_\mu) = \frac{1}{4\pi} \lim_{M_{\text{Reg}} \rightarrow \infty} \frac{M_{\text{Reg}}}{|M_{\text{Reg}}|} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \frac{1}{2G} (A_\mu - \partial_\mu \phi)^2, \quad (6.1)$$

where M_{Reg} is the mass of the Pauli-Villars regulator. Now an emphasis is in order on an important role of the fictitious NG boson field ϕ which is composed of the topologically nontrivial sector Θ and the smooth NG degree η for a given topological sector Θ as mentioned in the previous section. If we neglect the topological sector and consider only the smooth NG boson mode η , then the nonderivative gauge mass term dominates in the long range physics and the theory remains just in the topologically trivial sector of Chern-Simons Higgs phase which is governed by a parity-violating helicity one photon with mass $2\pi/G$ [35]. Inclusion of the topological sector under the

guiding principle of the HLS gives rise to the generation of CS vortices and realizes an anyonic phase[36]. We may recall that the addition of the CS term to the (2+1)-dimensional QED changed the structure of phase transition to the first-order one [37] and this theory has also been a model describing anyonic superconductivity. Then the subject of dynamical symmetry breaking in massless Thirring model for an odd number of fermions may generate intriguing results.

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Note added

After completion of our work we have received a recent paper by S. Hands, “ $O(1/N_f)$ corrections to the Thirring Model in $2 < d < 4$ ”, Wales Preprint, SWAT/94/47 (Nov., 1994), which addresses the problem of the renormalizability but not the dynamical mass generation of the fermion. This paper has the same difficulties as those in the previous papers which are resolved by the introduction of the hidden local symmetry as we mentioned in the text.

Appendix. BRS Invariance in the Nonlocal Gauge

In this Appendix we prove that the gauge fixing term Eq.(2.12) is BRS invariant even for the nonlocal gauge. It might be nontrivial to see that the nonlocal gauge fixing at Lagrangian level actually works.

According to the HLS transformations Eq.(2.5), the BRS transformation for each field is given by

$$\begin{aligned}
\delta_B \psi_a(x) &= \frac{i}{\sqrt{N}} c(x) \psi_a(x), \\
\delta_B A_\mu(x) &= \partial_\mu c(x), \\
\delta_B \phi(x) &= \frac{1}{\sqrt{N}} c(x), \\
\delta_B c(x) &= 0, \\
\delta_B \bar{c}(x) &= iB(x), \\
\delta_B B(x) &= 0,
\end{aligned} \tag{A.1}$$

where $c(x)$ and $\bar{c}(x)$ are ghost fields, and $B(x)$ is the so-called Nakanishi-Lautrap field. Moreover, it is well known that the operator δ_B is nilpotent:

$$\delta_B^2 = 0. \tag{A.2}$$

Following the text book procedure, we have the gauge fixing term plus Fadeev-Popov ghost term:

$$\mathcal{L}_{\text{GF+FP}} = -i\delta_B (\bar{c}f[A, \phi, c, \bar{c}, B]). \tag{A.3}$$

Without knowing explicit form of $f[A, \dots]$, we easily see from the nilpotency, Eq.(A.2), that $\mathcal{L}_{\text{GF+FP}}$ is BRS-invariant.

Now we show that the nonlocal R_ξ gauge fixing term (2.10) is obtained, if we take $f[A, \dots]$ as

$$\begin{aligned}
f[A, \dots] &= \partial_\mu A^\mu + \sqrt{N} \frac{\xi(\partial^2)}{G} \phi + \frac{1}{2} \xi(\partial^2) B \\
&= F[A] + \frac{1}{2} \xi(\partial^2) B,
\end{aligned} \tag{A.4}$$

with $F[A]$ being the same as Eq.(2.9). In fact, substituting Eq.(A.4) into Eq.(A.3), we find that $\mathcal{L}_{\text{GF+FP}}$ takes the form:

$$\mathcal{L}_{\text{GF}} = B \left\{ F[A] + \frac{1}{2} \xi(\partial^2) B \right\}, \quad (\text{A.5})$$

$$\mathcal{L}_{\text{FP}} = i\bar{c} \left(\square + \frac{\xi(\partial^2)}{G} \right) c. \quad (\text{A.6})$$

As a usual result for the Abelian gauge symmetry, FP ghost decouples from the system completely. We then translate $B(x) = B'(x) - \xi^{-1}(\partial^2)F[A]$ in Eq.(A.5) and integrate out $B'(x)$, arriving finally at

$$\mathcal{L}'_{\text{GF}} = -\frac{1}{2}F[A] \frac{1}{\xi(\partial^2)} F[A], \quad (\text{A.7})$$

where we have used the following identity:

$$\int d^D x f(x) \xi(\partial^2) g(x) = \int d^D x \left(\xi(\partial^2) f(x) \right) g(x), \quad (\text{A.8})$$

which can easily be proved in momentum space.

Eq.(A.7) is nothing but the nonlocal R_ξ gauge fixing term in Eq.(2.10).

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